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Solutions for certain classes of the Riccati differential equation

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Abstract

We derive some analytic closed-form solutions for a class of Riccati equations $y'(x) - \lambda_0(x)y(x) \pm y^2(x) = \pm s_0(x)$ where $\lambda_0(x)$, $s_0(x)$ are C^∞ -functions. We show that if $\delta_n = \lambda_n s_{n-1} - \lambda_{n-1} s_n = 0$, where $\lambda_n = \lambda'_{n-1} + s_{n-1} + \lambda_0 \lambda_{n-1}$ and $s_n = s'_{n-1} + s_0 \lambda_{n-1}$, $n = 1, 2, \dots$, then the Riccati equation has a solution given by $y(x) = \mp s_{n-1}(x)/\lambda_{n-1}(x)$. Extension to the generalized Riccati equation $y'(x) + P(x)y(x) + Q(x)y^2(x) = R(x)$ is also investigated.

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1. Introduction

The present authors have recently introduced an iterative technique [1], known as the asymptotic iteration method (AIM), for the exact and approximate solution of the second-order homogeneous differential equation

$$u''(x) = \lambda_0(x)u'(x) + s_0(x)u(x), \quad (1)$$

where $\lambda_0(x)$ and $s_0(x)$ are C^∞ -differentiable functions. It was shown that if for sufficiently large $n > 0$,

$$\frac{s_n}{\lambda_n} = \frac{s_{n-1}}{\lambda_{n-1}} = \alpha, \quad (2)$$

where

$$\lambda_n = \lambda'_{n-1} + s_{n-1} + \lambda_0 \lambda_{n-1} \quad \text{and} \quad s_n = s'_{n-1} + s_0 \lambda_{n-1}, \quad n = 1, 2, \dots \quad (3)$$

then

$$y(x) = \exp\left(-\int^x \alpha(t) dt\right) \left[C_2 + C_1 \int^x \exp\left(\int^t (\lambda_0(\tau) + 2\alpha(\tau)) d\tau\right) dt \right] \quad (4)$$

is the general solution of the differential equation (1). Saad *et al* [2] proved that the termination condition (2) is necessary and sufficient for the differential equation (1) to have polynomial-type solutions. Using the termination condition (2), the authors were able to show that the classical differential equation of Laguerre, Hermite, Legendre, Jacobi, etc obey this simple criterion. Continuing the work started in [2], in the present paper we derive some analytic closed-form solutions for different classes of the nonlinear first-order Riccati equation:

$$y' + P(x)y + Q(x)y^2 = R(x). \quad (5)$$

The principal idea comes from the fact that every Riccati equation reduces to a second-order linear differential equation by some suitable transformation [3]. Using this idea along with the asymptotic iteration method [1, 2], we construct closed-form solutions for different classes of the Riccati equation (5) under certain conditions on the functions $P(x)$, $Q(x)$ and $R(x)$. The nonlinear Riccati equation is of great interest for many applications to mathematical physics. It is well known that to find the general solution of the Riccati equation it is enough to know one nontrivial particular solution [3–5]. The paper is organized as follows: in the next section we discuss two simple cases $y' - \lambda_0(x)y \pm y^2 = \pm s_0(x)$, where $\lambda_0(x)$ and $s_0(x)$ are differentiable functions. In section 3, we discuss the generalized Riccati equation (5). We show that for certain relations connecting the functions $P(x)$, $Q(x)$ and $R(x)$, we can construct many closed-form solutions of the Riccati equation. Selections of results are presented in the tables, which can easily be extended.

2. Two simple cases

The following theorem provides condition for the solvability of a certain class of (5).

Theorem 1. *Given λ_0 and s_0 in $C^\infty(a, b)$, the Riccati equation*

$$y' - \lambda_0(x)y + y^2 = s_0(x) \quad (6)$$

has a solution

$$y_n(x) = -\frac{s_{n-1}(x)}{\lambda_{n-1}(x)} \quad (7)$$

if for some $n > 0$, $\delta_n = \lambda_n s_{n-1} - \lambda_{n-1} s_n = 0$ where λ_n and s_n satisfy the recurrence relations (3).

The validity of this theorem can be easily verified by direct substitution of (7) into (6) and the application of the termination condition (2) and the AIM sequence (3). However, a more constructive proof can be established by substituting $y = \frac{u'}{u}$ in equation (6) to obtain the well-known second-order differential equation (1). Using the termination condition (2), we have for $\delta_n = \lambda_n s_{n-1} - \lambda_{n-1} s_n = 0$, where λ_n and s_n are given by (3), that the differential equation (1) has a solution [2]

$$u(x) = \exp\left(-\int^x \frac{s_{n-1}(\tau)}{\lambda_{n-1}(\tau)} d\tau\right), \quad (8)$$

which implies $y(x) = \frac{u'(x)}{u(x)} = -\frac{s_{n-1}(x)}{\lambda_{n-1}(x)}$, as required.

Example 1. Consider the Riccati differential equation

$$y'(x) + \frac{((m-a)x^2 + (2cm-1)x - c)}{(ax^3 + bx^2 + cx)}y(x) + y^2(x) = -\frac{(-2mx+1)}{(ax^3 + bx^2 + cx)}.$$

With $\lambda_0(x) = -\frac{(m-a)x^2+(2cm-1)x-c}{(ax^3+bx^2+cx)}$ and $s_0(x) = -\frac{(-2mx+1)}{(ax^3+bx^2+cx)}$, we can easily show $\delta_2 = \lambda_2 s_1 - \lambda_1 s_2 = 0$, where

$$\lambda_2(x) = -\frac{((m+a)x^2+(4cm+2b-1)(x+c))(m(m+3a)x^2+(2m^2c-2(1-b)m-4a)x-3b-3cm+1)}{x(ax^2+bx+c)^3}$$

and

$$s_2(x) = \frac{(2(m+a)x+4cm+2b-1)(m(m+3a)x^2+(2m^2c-2(1-b)m-4a)x-3b-3cm+1)}{x(ax^2+bx+c)^3}.$$

Thus a solution is

$$y(x) = \frac{s_1(x)}{\lambda_1(x)} = \frac{2(m+a)x+4cm+2b-1}{(m+a)x^2+(4cm+2b-1)(x+c)}.$$

In table 1, we present some closed-form solutions for different classes of Riccati equation, obtained by direct applications of theorem 1. In this table, we use the generalized hypergeometric series ${}_pF_q(\alpha_1, \alpha_2, \dots, \alpha_p; \beta_1, \beta_2, \dots, \beta_q; x)$ defined by [6]

$${}_pF_q(\alpha_1, \alpha_2, \dots, \alpha_p; \beta_1, \beta_2, \dots, \beta_q; x) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k (\alpha_2)_k \dots (\alpha_p)_k x^k}{(\beta_1)_k (\beta_2)_k \dots (\beta_q)_k k!} \quad (9)$$

where p and q are nonnegative integers and no $\beta_k, k = 1, 2, \dots, q$ is zero or a negative integer. Clearly, (9) includes the special cases of the confluent hypergeometric function ${}_1F_1$ and the classical ‘Gaussian’ hypergeometric function ${}_2F_1$. The Pochhammer symbol $(\alpha)_k$ is defined in terms of Gamma function as

$$(\alpha)_k = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}, \quad k = 0, 1, 2, \dots \quad (10)$$

If α is a negative integer $-n$, we have

$$(-n)_k = \begin{cases} \frac{(-1)^k n!}{(n-k)!} & 0 \leq k \leq n \\ 0 & k > n \end{cases} \quad (11)$$

in which case, the generalized hypergeometric series reduces to a polynomial of degree n in its variable x .

Theorem 2. Given λ_0 and s_0 in $C^\infty(a, b)$, the Riccati equation

$$y' - \lambda_0(x)y - y^2 = -s_0(x) \quad (12)$$

has a solution

$$y_n(x) = \frac{s_{n-1}(x)}{\lambda_{n-1}(x)} \quad (13)$$

if for some $n > 0$, $\delta_n = \lambda_n s_{n-1} - \lambda_{n-1} s_n = 0$ where λ_n and s_n satisfy the recurrence relations (3).

Proof. We substitute $y = -\frac{u'}{u}$ in (12) and thereby obtain the differential equation (1) which, by using AIM, yields $\frac{u'}{u} = -\frac{s_{n-1}}{\lambda_{n-1}}$. Therefore the solution of the Riccati equation (12) is given by (13). \square

Table 1. Closed-form solutions for the Riccati differential equation $y' - \lambda_0(x)y + y^2 = s_0(x)$ by theorem 1.

Riccati equation	$y_n, n = 0, 1, 2, \dots$
$y' - 2xy + y^2 = -4n$	$y_n = -4nx \frac{{}_1F_1(-n+1; \frac{3}{2}; x^2)}{{}_1F_1(-n; \frac{1}{2}; x^2)}$
$y' - 2xy + y^2 = -2(2n+1)$	$y_n = \frac{1}{x} - \frac{4nx}{3} \frac{{}_1F_1(-n+1; \frac{5}{2}; x^2)}{{}_1F_1(-n; \frac{3}{2}; x^2)}$
$y' - (ax+b)y + y^2 = -2na$	$y_n = -2n(ax+b) \frac{{}_1F_1(-n+1; \frac{3}{2}; \frac{(ax+b)^2}{2a})}{{}_1F_1(-n; \frac{1}{2}; \frac{(ax+b)^2}{2a})}$
$y' - (ax+b)y + y^2 = -(2n+1)a$	$y_n = \frac{a}{ax+b} - \frac{2n}{3}(ax+b) \frac{{}_1F_1(-n+1; \frac{5}{2}; \frac{(ax+b)^2}{2a})}{{}_1F_1(-n; \frac{3}{2}; \frac{(ax+b)^2}{2a})}$
$y' - (b - \frac{c}{x})y + y^2 = -\frac{nb}{x}$	$y_n = -\frac{nb}{c} \frac{{}_1F_1(-n+1; c+1; bx)}{{}_1F_1(-n; c; bx)}$
$y' - \frac{(b-n+1)x-c}{x(1-x)}y + y^2 = -\frac{nb}{x(1-x)}$	$y_n = -\frac{nb}{c} \frac{{}_2F_1(-n+1, b+1; c+1; x)}{{}_2F_1(-n, b; c; x)}$
$y' - \frac{(-2n+1)x-c}{x(1-x)}y + y^2 = -\frac{n^2}{x(1-x)}$	$y_n = \frac{n^2}{c} \frac{{}_2F_1(-n+1, -n+1; c+1; x)}{{}_2F_1(-n, -n; c; x)}$
$y' - \frac{x}{(1-x^2)}y + y^2 = -\frac{n^2}{1-x^2}$	$y_n = n^2 \frac{{}_2F_1(-n+1, n+1; \frac{3}{2}; \frac{1-x}{2})}{{}_2F_1(-n, n; \frac{1}{2}; \frac{1-x}{2})}$
$y' - \frac{2x}{(1-x^2)}y + y^2 = -\frac{n(n+1)}{(1-x^2)}$	$y_n = \frac{n(n+1)}{2} \frac{{}_2F_1(-n+1, n+2; 2; \frac{1-x}{2})}{{}_2F_1(-n, n+1; 1; \frac{1-x}{2})}$
$y' - \frac{3x}{(1-x^2)}y + y^2 = -\frac{n(n+2)}{(1-x^2)}$	$y_n = \frac{n(n+2)}{3} \frac{{}_2F_1(-n+1, n+3; \frac{5}{2}; \frac{1-x}{2})}{{}_2F_1(-n, n+2; \frac{3}{2}; \frac{1-x}{2})}$
$y' - \frac{ax}{(1-x^2)}y + y^2 = -\frac{n(n+a-1)}{(1-x^2)}$	$y_n = \frac{n(n+a-1)}{a} \frac{{}_2F_1(-n+1, n+a; \frac{a}{2}+1; \frac{1-x}{2})}{{}_2F_1(-n, n+a-1; \frac{a}{2}; \frac{1-x}{2})}$
$y' - \frac{(a+b+2)x-b+a}{(1-x^2)}y + y^2 = -\frac{n(n+a+b+1)}{(1-x^2)}$	$y_n = \frac{n(n+a+b+1)}{2(a+1)} \frac{{}_2F_1(-n+1, n+a+b+2; a+2; \frac{1-x}{2})}{{}_2F_1(-n, n+a+b+1; a+1; \frac{1-x}{2})}$
$y' - \frac{(1+2k)x}{(1-x^2)}y + y^2 = -\frac{n(n+2k)}{(1-x^2)}$	$y_n = \frac{n(n+2k)}{2} \frac{{}_2F_1(-n+1, n+2k+1; k+\frac{3}{2}; \frac{1-x}{2})}{{}_2F_1(-n, n+2k; k+\frac{1}{2}; \frac{1-x}{2})}$
$y' - \frac{2(1+k)x}{(1-x^2)}y + y^2 = -\frac{n(n+2k+1)}{(1-x^2)}$	$y_n = \frac{n(n+2k+1)}{2(k+1)} \frac{{}_2F_1(-n+1, n+2k+2; k+2; \frac{1-x}{2})}{{}_2F_1(-n, n+2k+1; k+1; \frac{1-x}{2})}$
$y' + \frac{2(x+1)}{x^2}y + y^2 = \frac{n(n+1)}{x^2}$	$y_n = \frac{n(n+1)}{2} \frac{{}_2F_0(-n+1, n+2; -; -\frac{x}{2})}{{}_2F_0(-n, n+1; -; -\frac{x}{2})}$
$y' + \frac{(ax+b)}{x^2}y + y^2 = \frac{n(n+a-1)}{x^2}$	$y_n = \frac{n(n+a-1)}{b} \frac{{}_2F_0(-n+1, n+a; -; -\frac{x}{b})}{{}_2F_0(-n, n+a-1; -; -\frac{x}{b})}$

Example 2. Consider the Riccati equation

$$y' - \left(3ax + \frac{1}{x}\right)y - y^2 = -a^2,$$

where a is a constant. Direct computation yields

$$\delta_{2n} = \prod_{k=1}^n (a + 6k) = 0.$$

Consequently, we have

$$y_2 = -\frac{2}{x}, \quad y_4 = -\frac{2(18x^2 + 1)}{x(1 + 9x^2)}, \quad y_6 = -\frac{2(729x^4 + 108x^2 + 2)}{x(243x^4 + 54x^2 + 2)}, \dots$$

In table 2, we present some closed-form solutions for different classes of the Riccati equation, as direct applications of theorem 2.

Table 2. Closed-form solutions for the Riccati differential equation $y' - \lambda_0(x)y - y^2 = -s_0(x)$ by theorem 2.

Riccati equation	$y_n, n = 0, 1, 2, \dots$
$y' - 2xy - y^2 = 4n$	$y_n = 4nx \frac{{}_1F_1(-n+1; \frac{3}{2}; x^2)}{{}_1F_1(-n; \frac{1}{2}; x^2)}$
$y' - 2xy - y^2 = 2(2n+1)$	$y_n = -\frac{1}{x} + \frac{4nx}{3} \frac{{}_1F_1(-n+1; \frac{5}{2}; x^2)}{{}_1F_1(-n; \frac{3}{2}; x^2)}$
$y' - (ax+b)y - y^2 = 2na$	$y_n = 2n(ax+b) \frac{{}_1F_1(-n+1; \frac{3}{2}; \frac{(ax+b)^2}{2a})}{{}_1F_1(-n; \frac{1}{2}; \frac{(ax+b)^2}{2a})}$
$y' - (ax+b)y - y^2 = (2n+1)a$	$y_n = -\frac{a}{ax+b} + \frac{2n}{3}(ax+b) \frac{{}_1F_1(-n+1; \frac{5}{2}; \frac{(ax+b)^2}{2a})}{{}_1F_1(-n; \frac{3}{2}; \frac{(ax+b)^2}{2a})}$
$y' - (b - \frac{c}{x})y - y^2 = \frac{nb}{x}$	$y_n = \frac{nb}{c} \frac{{}_1F_1(-n+1; c+1; bx)}{{}_1F_1(-n; c; bx)}$
$y' - \frac{(b-n+1)x-c}{x(1-x)}y - y^2 = \frac{nb}{x(1-x)}$	$y_n = \frac{nb}{c} \frac{{}_2F_1(-n+1, b+1; c+1; x)}{{}_2F_1(-n, b; c; x)}$
$y' - \frac{(-2n+1)x-c}{x(1-x)}y - y^2 = \frac{n^2}{x(1-x)}$	$y_n = -\frac{n^2}{c} \frac{{}_2F_1(-n+1, -n+1; c+1; x)}{{}_2F_1(-n, -n; c; x)}$
$y' - \frac{x}{(1-x^2)}y - y^2 = \frac{n^2}{1-x^2}$	$y_n = -n^2 \frac{{}_2F_1(-n+1, n+1; \frac{3}{2}; \frac{1-x}{1+x})}{{}_2F_1(-n, n; \frac{1}{2}; \frac{1-x}{1+x})}$
$y' - \frac{2x}{(1-x^2)}y - y^2 = \frac{n(n+1)}{(1-x^2)}$	$y_n = -\frac{n(n+1)}{2} \frac{{}_2F_1(-n+1, n+2; 2; \frac{1-x}{1+x})}{{}_2F_1(-n, n+1; 1; \frac{1-x}{1+x})}$
$y' - \frac{3x}{(1-x^2)}y - y^2 = \frac{n(n+2)}{(1-x^2)}$	$y_n = -\frac{n(n+2)}{3} \frac{{}_2F_1(-n+1, n+3; \frac{5}{2}; \frac{1-x}{1+x})}{{}_2F_1(-n, n+2; \frac{3}{2}; \frac{1-x}{1+x})}$
$y' - \frac{ax}{(1-x^2)}y - y^2 = \frac{n(n+a-1)}{(1-x^2)}$	$y_n = -\frac{n(n+a-1)}{a} \frac{{}_2F_1(-n+1, n+a; \frac{a}{2}+1; \frac{1-x}{1+x})}{{}_2F_1(-n, n+a-1; \frac{a}{2}; \frac{1-x}{1+x})}$
$y' - \frac{(a+b+2)x-b+a}{(1-x^2)}y - y^2 = \frac{n(n+a+b+1)}{(1-x^2)}$	$y_n = -\frac{n(n+a+b+1)}{2(a+1)} \frac{{}_2F_1(-n+1, n+a+b+2, a+2; \frac{1-x}{1+x})}{{}_2F_1(-n, n+a+b+1, a+1; \frac{1-x}{1+x})}$
$y' - \frac{(1+2k)x}{(1-x^2)}y - y^2 = \frac{n(n+2k)}{(1-x^2)}$	$y_n = -\frac{n(n+2k)}{2} \frac{{}_2F_1(-n+1, n+2k+1; k+\frac{3}{2}; \frac{1-x}{1+x})}{{}_2F_1(-n, n+2k; k+\frac{1}{2}; \frac{1-x}{1+x})}$
$y' - \frac{2(1+k)x}{(1-x^2)}y - y^2 = \frac{n(n+2k+1)}{(1-x^2)}$	$y_n = -\frac{n(n+2k+1)}{2(k+1)} \frac{{}_2F_1(-n+1, n+2k+2; k+2; \frac{1-x}{1+x})}{{}_2F_1(-n, n+2k+1; k+1; \frac{1-x}{1+x})}$
$y' + \frac{2(x+1)}{x^2}y - y^2 = -\frac{n(n+1)}{x^2}$	$y_n = -\frac{n(n+1)}{2} \frac{{}_2F_0(-n+1, n+2; -; -\frac{x}{2})}{{}_2F_0(-n, n+1; -; -\frac{x}{2})}$
$y' + \frac{(ax+b)}{x^2}y - y^2 = -\frac{n(n+a-1)}{x^2}$	$y_n = -\frac{n(n+a-1)}{b} \frac{{}_2F_0(-n+1, n+a; -; -\frac{x}{b})}{{}_2F_0(-n, n+a-1; -; -\frac{x}{b})}$

3. Generalized Riccati equation

The differential equation,

$$\frac{dy}{dx} + P(x)y + Q(x)y^2 = R(x), \quad (14)$$

is known as the generalized Riccati equation. A number of transformations exist for changing this Riccati equation to a second-order linear homogeneous equation (and vice versa). Some of these transformations are summarized in the table 3.

Theorem 3. The Riccati equation $y' + P(x)y + Q(x)y^2 = R(x)$, has a solution

$$y(x) = -\frac{s_{n-1}(x)}{Q(x)\lambda_{n-1}(x)} \quad (15)$$

where

$$\lambda_0 = \frac{Q'}{Q} - P \quad \text{and} \quad s_0 = QR \quad (16)$$

and for $n > 0$, λ_n and s_n are given by $\lambda_n = \lambda'_{n-1} + s_{n-1} + \lambda_0\lambda_{n-1}$ and $s_n = s'_{n-1} + s_0\lambda_{n-1}$, $n = 1, 2, \dots$ if for some $n > 0$, $\delta_n = \lambda_n s_{n-1} - \lambda_{n-1} s_n = 0$.

Table 3. Methods for transforming the generalized Riccati differential equation $y' + P(x)y + Q(x)y^2 = R(x)$ to a second-order homogeneous differential equation.

Transformation	Resulting equation	Reference
$y = \frac{u'}{Qu}$	$u'' = \left(\frac{Q'}{Q} - P\right)u' + QRu$	[7]
$y = \frac{Ru}{u'}$	$u'' = \left(P + \frac{R'}{R}\right)u' + QRu$	[8]
$y = \frac{Ru}{u'+Pu}$	$u'' = \left(\frac{R'}{R} - P\right)u' + R\left(Q - \left(\frac{P}{R}\right)'\right)u$	[9]
$y = \frac{u'-Pu}{Qu}$	$u'' = \left(P + \frac{Q'}{Q}\right)u' + Q\left(\left(\frac{P}{Q}\right)' + R\right)u$	[9]

Proof. We substitute $y = \frac{u'}{Qu}$ in the Riccati equation and thereby obtain the second-order differential equation $u'' = \left(\frac{Q'}{Q} - P\right)u' + QRu$. By using AIM with $\lambda_0 = \frac{Q'}{Q} - P$ and $s_0 = QR$, we have $\frac{u'}{u} = -\frac{s_{n-1}}{\lambda_{n-1}}$ if λ_0 and s_0 , along with AIM sequence (3); thus we satisfy the termination condition $\delta_n = 0$. \square

Example 3. For the Riccati differential equation

$$y'(x) + y(x) + e^{\frac{3}{4}x^4+x} y^2(x) = -27x^2 e^{-\frac{3}{4}x^4-x},$$

with $\lambda_0(x) = \frac{Q'}{Q} - P = 3x^3$ and $s_0(x) = QR = -27x^2$, we can easily show $\delta_9 = \lambda_9 s_8 - \lambda_8 s_9 = 0$. Thus a solution of the given differential equation is given by

$$y(x) = -\frac{s_8(x)}{Q(x)\lambda_8(x)} = \frac{9x^8 - 30x^4 + 5}{x e^{\frac{3}{4}x^4+x} (x^8 - 6x^4 + 5)}.$$

Example 4. Consider the Riccati differential equation

$$y'(x) - \frac{b}{x}y(x) - ax^n y^2(x) = \frac{c}{x^{n+2}}.$$

Here $Q = -ax^n$, $P = -\frac{b}{x}$ and $R = \frac{c}{x^{n+2}}$, we have $\lambda_0(x) = \frac{Q'}{Q} - P = \frac{n+b}{x}$ and $s_0(x) = QR = -\frac{ac}{x^2}$, we can easily show

$$\delta_1 = \lambda_1 s_0 - \lambda_0 s_1 = \frac{ac(-n-b+ac)}{x^4} = 0 \quad \text{if} \quad ac - (n+b) = 0$$

$$\Rightarrow y_1(x) = -\frac{s_0}{Q\lambda_0} = -\frac{1}{ax^{n+1}}$$

$$\delta_2 = \lambda_2 s_1 - \lambda_1 s_2 = \frac{ac(-n-b+ac)(-2b+2-2n+ac)}{x^6} = 0 \quad \text{if} \quad ac - 2(n+b) + 2 = 0$$

$$\Rightarrow y_2(x) = -\frac{s_1}{Q\lambda_1} = -\frac{2}{ax^{n+1}}$$

$$\delta_3 = \lambda_3 s_2 - \lambda_2 s_3 = \frac{ac(-n-b+ac)(-2b+2-2n+ac)(-3b+6-3n+ac)}{x^8}$$

$$= 0 \quad \text{if} \quad ac - 3(n+b) + 6 = 0$$

$$\Rightarrow y_3(x) = -\frac{s_2}{Q\lambda_2} = -\frac{3}{ax^{n+1}}$$

and so on. It is clear that $\delta_m = 0$ for $m = 1, 2, \dots$, if $ac - m(n+b) + m(m-1) = 0$, and the solution is given by

$$y_m(x) = -\frac{m}{ax^{n+1}} \quad \text{for} \quad m = 1, 2, \dots$$

This result is expected since the corresponding second-order differential equation is the well-known Cauchy–Euler differential equation.

There are some interesting applications that follow from theorem 3.

- If we know that λ_0 and s_0 satisfy the termination condition (2), we can solve (16) for $Q(x)$ and $R(x)$ as

$$\begin{aligned} Q(x) &= \exp\left(\int^x (\lambda_0(\tau) + P(\tau)) d\tau\right), \\ R(x) &= s_0(x) \exp\left(-\int^x (\lambda_0(\tau) + P(\tau)) d\tau\right), \end{aligned} \quad (17)$$

where $P(x)$ is an arbitrary integrable function. Thus, the Riccati equation

$$y' + P(x)y + e^{\int^x (\lambda_0(\tau) + P(\tau)) d\tau} y^2 = s_0(x) e^{-\int^x (\lambda_0(\tau) + P(\tau)) d\tau} \quad (18)$$

has the particular solution

$$y(x) = -\frac{s_{n-1}(x)}{\lambda_{n-1}(x)} e^{-\int^x (\lambda_0(\tau) + P(\tau)) d\tau}. \quad (19)$$

In order to illustrate this idea, we know [2] for $\lambda_0 = 2x$ and $s_0 = -2k$ that $\delta_n = 0$ for $n = k, k = 0, 2, 4, \dots$. Thus, with $Q(x) = e^{x^2 + \int^x P(\tau) d\tau}$ and $R(x) = -4n e^{-x^2 - \int^x P(\tau) d\tau}$, the differential equation

$$y' + P(x)y + e^{x^2 + \int^x P(\tau) d\tau} y^2 = -4n e^{-x^2 - \int^x P(\tau) d\tau},$$

has the particular solution

$$y_n(x) = -4nx e^{-x^2 - \int^x P(\tau) d\tau} \frac{{}_1F_1(-n+1; \frac{3}{2}; x^2)}{{}_1F_1(-n; \frac{1}{2}; x^2)} \quad \text{for } n = 0, 1, 2, \dots \quad (20)$$

Furthermore, the differential equation

$$y' + P(x)y + e^{x^2 + \int^x P(\tau) d\tau} y^2 = -2(2n+1) e^{-x^2 - \int^x P(\tau) d\tau}, \quad (21)$$

has the solution

$$y_n = e^{-x^2 - \int^x P(\tau) d\tau} \left(\frac{1}{x} - \frac{4nx}{3} \frac{{}_1F_1(-n+1; \frac{5}{2}; x^2)}{{}_1F_1(-n; \frac{3}{2}; x^2)} \right), \quad n = 0, 1, 2, \dots \quad (22)$$

Note that, in the case of $P(x) = -\lambda_0(x)$, we recover (6). In table 4, we present some closed-form solutions for different classes of Riccati equation (18) for known $\lambda_0(x)$ and $s_0(x)$ [2].

- The Riccati equation

$$y' + \left(\frac{s'_0(x)}{s_0(x)} - \frac{R'(x)}{R(x)} - \lambda_0(x) \right) y + \frac{s_0(x)}{R(x)} y^2 = R(x) \quad (23)$$

has the particular solution

$$y(x) = -\frac{R(x) s_{n-1}(x)}{s_0(x) \lambda_{n-1}(x)} \quad (24)$$

if λ_0 and s_0 , along with the AIM sequence (3), satisfies the termination condition $\delta_n = 0$.

- For the Riccati equation $y' + P(x)y + Q(x)y^2 = R(x)$, if

$$\frac{Q'}{Q} - P + xQR = 0, \quad (25)$$

then $y = \frac{1}{xQ(x)}$ is a particular solution. For example, the Riccati differential equation $y'(x) - xf(x)y - y^2 = f(x)$ for arbitrary differentiable function f has a solution given by $y = -1/x$. This follows from the fact that, if $\lambda_0 = -xs_0$, then $\delta_1 = 0$, and the corresponding second-order differential equation has the solution $u = x$.

Table 4. Closed-form solutions for the Riccati differential equation $y' + P(x)y + Q(x)y^2 = R(x)$ by theorem 3. Here $P(x)$ is an arbitrary integrable function.

$Q(x)$	$R(x)$	$y_n, n = 0, 1, 2, \dots$
$e^{x^2+f^x P(\tau) d\tau}$	$-4n e^{-x^2-f^x P(\tau) d\tau}$	$-4nx e^{-x^2} \frac{{}_1F_1(-n+1; \frac{3}{2}; x^2)}{{}_1F_1(-n; \frac{1}{2}; x^2)} e^{-f^x P(\tau) d\tau}$
$e^{x^2+f^x P(\tau) d\tau}$	$-2(2n+1) e^{-x^2-f^x P(\tau) d\tau}$	$e^{-x^2} \left(\frac{1}{x} - \frac{4nx}{3} \frac{{}_1F_1(-n+1; \frac{5}{2}; x^2)}{{}_1F_1(-n; \frac{3}{2}; x^2)} \right) e^{-f^x P(\tau) d\tau}$
$\frac{e^x}{x} e^{f^x P(\tau) d\tau}$	$-n e^{-x} e^{-f^x P(\tau) d\tau}$	$-nx e^{-x} \frac{{}_1F_1(-n+1; 2; x)}{{}_1F_1(-n; 1; x)} e^{-f^x P(\tau) d\tau}$
$\frac{e^{bx}}{x^c} e^{f^x P(\tau) d\tau}$	$-nbx^{c-1} e^{-bx} e^{-f^x P(\tau) d\tau}$	$-\frac{nb}{c} x^c e^{-bx} \frac{{}_1F_1(-n+1; c+1; bx)}{{}_1F_1(-n; c; bx)} e^{-f^x P(\tau) d\tau}$
$\frac{(x-1)^{c+2n-1}}{x^c} e^{f^x P(\tau) d\tau}$	$\frac{-n^2 x^{c-1}}{(x-1)^{2n+c}} e^{-f^x P(\tau) d\tau}$	$\frac{n^2 x^c}{c(x-1)^{2n+c-1}} \frac{{}_2F_1(-n+1, -n+1; c+1; x)}{{}_2F_1(-n, -n; c; x)} e^{-f^x P(\tau) d\tau}$
$\frac{(x-1)^{c+n-b-1}}{x^c} e^{f^x P(\tau) d\tau}$	$\frac{nbx^{c-1}}{(x-1)^{n+c-b}} e^{-f^x P(\tau) d\tau}$	$-\frac{nbx^c}{c(x-1)^{c+n-1-b}} \frac{{}_2F_1(-n+1, b+1; c+1; x)}{{}_2F_1(-n, b; c; x)} e^{-f^x P(\tau) d\tau}$
$\frac{1}{\sqrt{x^2-1}} e^{f^x P(\tau) d\tau}$	$\frac{n^2}{\sqrt{x^2-1}} e^{-f^x P(\tau) d\tau}$	$n^2 \sqrt{x^2-1} \frac{{}_2F_1(-n+1, n+1; \frac{3}{2}, \frac{1-x}{2})}{{}_2F_1(-n, n; \frac{1}{2}, \frac{1-x}{2})} e^{-f^x P(\tau) d\tau}$
$\frac{1}{x^2-1} e^{f^x P(\tau) d\tau}$	$n(n+1) e^{-f^x P(\tau) d\tau}$	$\frac{1}{2} n(n+1)(x^2-1) e^{-f^x P(\tau) d\tau} \frac{{}_2F_1(-n+1, n+2; 2, \frac{1-x}{2})}{{}_2F_1(-n, n+1; 1, \frac{1-x}{2})}$
$\frac{1}{(x^2-1)^{3/2}} e^{f^x P(\tau) d\tau}$	$n(n+2)\sqrt{x^2-1} e^{-f^x P(\tau) d\tau}$	$\frac{n(n+2)}{3} (x^2-1)^{3/2} e^{-f^x P(\tau) d\tau} \frac{{}_2F_1(-n+1, n+3; \frac{5}{2}, \frac{1-x}{2})}{{}_2F_1(-n, n+2; \frac{3}{2}, \frac{1-x}{2})}$
$\frac{1}{(x^2-1)^{a/2}} e^{f^x P(\tau) d\tau}$	$n(n+a-1)(x^2-1)^{\frac{a}{2}-1} e^{-f^x P(\tau) d\tau}$	$\frac{n(n+a-1)}{a} (x^2-1)^{a/2} e^{-f^x P(\tau) d\tau} \frac{{}_2F_1(-n+1, n+a; \frac{a}{2}+1, \frac{1-x}{2})}{{}_2F_1(-n, n+a-1; \frac{a}{2}, \frac{1-x}{2})}$
$\frac{1}{(x^2-1)^{k+1}} e^{f^x P(\tau) d\tau}$	$n(n+2k+1)(x^2-1)^k e^{-f^x P(\tau) d\tau}$	$\frac{n(n+2k+1)}{2(k+1)} (x^2-1)^{k+1} e^{-f^x P(\tau) d\tau} \frac{{}_2F_1(-n+1, n+2k+2; k+2; \frac{1-x}{2})}{{}_2F_1(-n, n+2k+1; k+1; \frac{1-x}{2})}$
$\frac{1}{(x^2-1)^{k+1/2}} e^{f^x P(\tau) d\tau}$	$n(n+2k)(x^2-1)^{k-1/2} e^{-f^x P(\tau) d\tau}$	$\frac{n(n+2k)}{1+2k} (x^2-1)^{k+1/2} e^{-f^x P(\tau) d\tau} \frac{{}_2F_1(-n+1, n+2k+1; k+\frac{3}{2}; \frac{1-x}{2})}{{}_2F_1(-n, n+2k; k+\frac{1}{2}; \frac{1-x}{2})}$
$\frac{1}{x^2} e^{2/x} e^{f^x P(\tau) d\tau}$	$n(n+1) e^{-2/x} e^{-f^x P(\tau) d\tau}$	$\frac{n(n+1)}{2} x^2 e^{-2/x} e^{-f^x P(\tau) d\tau} \frac{{}_2F_0(-n+1, n+2; -, -\frac{x}{2})}{{}_2F_0(-n, n+1; -, -\frac{x}{2})}$
$\frac{1}{x^a} e^{b/x} e^{f^x P(\tau) d\tau}$	$n(n+a-1)x^{a-2} e^{-b/x} e^{-f^x P(\tau) d\tau}$	$\frac{n(n+a-1)}{b} x^a e^{-b/x} e^{-f^x P(\tau) d\tau} \frac{{}_2F_0(-n+1, n+a; -, -\frac{x}{b})}{{}_2F_0(-n, n+a-1; -, -\frac{x}{b})}$
$\frac{1}{(x-1)^{a+1}(x+1)^{b+1}} e^{f^x P(\tau) d\tau}$	$\frac{n(n+a+b+1)}{(x-1)^{-a}(x+1)^{-b}} e^{-f^x P(\tau) d\tau}$	$\frac{n(n+a+b+1)(x^2-1)}{2(a+1)(x-1)^{-a}(x+1)^{-b}} e^{-f^x P(\tau) d\tau} \frac{{}_2F_1(-n+1, n+a+b+2; a+2; \frac{1-x}{2})}{{}_2F_1(-n, n+a+b+1; a+1; \frac{1-x}{2})}$

By means of the transformation $y = \frac{Ru}{u'}$, the following theorem can easily be proved.

Theorem 4. *The Riccati differential equation $y' + P(x)y + Q(x)y^2 = R(x)$, has the particular solution*

$$y(x) = -\frac{R(x)\lambda_{n-1}(x)}{s_{n-1}(x)}, \quad (26)$$

if for some $n > 0$, $\delta_n = \lambda_n s_{n-1} - \lambda_{n-1} s_n = 0$ where $\lambda_0 = \frac{R'}{R} + P$ and $s_0 = QR$.

In table 5, we present closed-form solutions for different classes of Riccati equation, obtained as direct applications of theorem 4.

By means of the transformation $y = \frac{Ru}{u'+Pu}$, it becomes straightforward to prove the following theorem.

Theorem 5. *The Riccati differential equation $y' + P(x)y + Q(x)y^2 = R(x)$, has the particular solution*

$$y(x) = \frac{R(x)\lambda_{n-1}(x)}{-s_{n-1}(x) + P(x)\lambda_{n-1}(x)}, \quad (27)$$

if for some $n > 0$, $\delta_n = \lambda_n s_{n-1} - \lambda_{n-1} s_n = 0$ where $\lambda_0 = \frac{R'}{R} - P$ and $s_0 = R(Q - (\frac{P}{R})')$.

There are two immediate consequences of this theorem.

- For an arbitrary function $R(x)$, the Riccati equation

$$y' + \left(\frac{R'(x)}{R(x)} - \lambda_0(x) \right) y + \left(\frac{s_0(x)}{R(x)} + \left(\frac{R'(x)}{R^2(x)} - \frac{\lambda_0(x)}{R(x)} \right)' \right) y^2 = R(x) \quad (28)$$

has the particular solution

$$y_n(x) = \frac{R(x)\lambda_{n-1}(x)}{-s_{n-1}(x) + \left(\frac{R'(x)}{R(x)} - \lambda_0(x) \right) \lambda_{n-1}(x)} \quad \text{for } n = 0, 1, 2, \dots \quad (29)$$

if $\lambda_0(x)$ and $s_0(x)$ satisfy the termination condition (2), namely $\delta_n = \lambda_n s_{n-1} - \lambda_{n-1} s_n = 0$ for $n = 0, 1, 2, \dots$. Note that the iteration sequence (3) can start with $n = 0$ for $\lambda_{-1} = 1$ and $s_{-1} = 0$. In table 6, we exhibit closed-form solutions for the Riccati equation (14).

- For an arbitrary function $P(x)$, the Riccati equation

$$y' + P(x)y + \left[s_0(x) e^{-\int^x (\lambda_0(\tau)+P(\tau)) d\tau} + \left(P(x) e^{-\int^x (\lambda_0(\tau)+P(\tau)) d\tau} \right)' \right] y^2 = e^{\int^x (\lambda_0(\tau)+P(\tau)) d\tau} \quad (30)$$

has the particular solution

$$y_n = \frac{\lambda_{n-1}(x) e^{\int^x (\lambda_0(\tau)+P(\tau)) d\tau}}{-s_{n-1}(x) + P(x)\lambda_{n-1}(x)} \quad \text{for } n = 0, 1, 2, \dots \quad (31)$$

if $\lambda_0(x)$ and $s_0(x)$ satisfy the termination condition (2), namely $\delta_n = \lambda_n s_{n-1} - \lambda_{n-1} s_n = 0$ for $n = 0, 1, 2, \dots$.

By means of the transformation $y = \frac{u'-Pu}{Qu}$, it then becomes straightforward to prove the following theorem.

Theorem 6. *The Riccati differential equation $\frac{dy}{dx} + P(x)y + Q(x)y^2 = R(x)$ has the particular solution*

$$y(x) = \frac{-s_{n-1}(x) - P(x)\lambda_{n-1}(x)}{Q(x)\lambda_{n-1}(x)}, \quad (32)$$

if for some $n > 0$, $\delta_n = \lambda_n s_{n-1} - \lambda_{n-1} s_n = 0$ where $\lambda_0 = P + \frac{Q'}{Q}$ and $s_0 = Q\left(\left(\frac{P}{Q}\right)' + R\right)$.

Table 5. Closed-form solutions for the Riccati differential equation $y' + P(x)y + Q(x)y^2 = R(x)$ by theorem 4. Here $P(x)$ is an arbitrary integrable function.

$Q(x)$	$R(x)$	$y_n, n = 1, 2, \dots$
$-4n e^{-x^2+f^x} P(\tau) \, d\tau$	$e^{x^2-f^x} P(\tau) \, d\tau$	$-\frac{1}{4nx} e^{x^2} \frac{{}_1F_1(-n; \frac{1}{2}; x^2)}{{}_1F_1(-n+1; \frac{3}{2}; x^2)} e^{-f^x} P(\tau) \, d\tau$
$-2(2n+1) e^{-x^2+f^x} P(\tau) \, d\tau$	$e^{x^2-f^x} P(\tau) \, d\tau$	$\frac{x e^{x^2} {}_1F_1(-n; \frac{3}{2}; x^2)}{{}_1F_1(-n; \frac{3}{2}; x^2) - \frac{4}{3} x^2 n {}_1F_1(-n+1; \frac{5}{2}; x^2)} e^{-f^x} P(\tau) \, d\tau$
$-2na e^{-\frac{ax^2}{2}-bx+f^x} P(\tau) \, d\tau$	$e^{\frac{ax^2}{2}+bx-f^x} P(\tau) \, d\tau$	$-\frac{e^{\frac{ax^2}{2}+bx}}{2n(ax+b)} \frac{{}_1F_1(-n; \frac{1}{2}; \frac{(ax+b)^2}{2a})}{{}_1F_1(-n+1; \frac{3}{2}; \frac{(ax+b)^2}{2a})} e^{-f^x} P(\tau) \, d\tau$
$-(2n+1)a e^{-\frac{ax^2}{2}-bx+f^x} P(\tau) \, d\tau$	$e^{\frac{ax^2}{2}+bx-f^x} P(\tau) \, d\tau$	$\frac{(ax+b) e^{\frac{ax^2}{2}+bx} {}_1F_1(-n; \frac{3}{2}; \frac{(ax+b)^2}{2a})}{a {}_1F_1(-n, \frac{3}{2}, \frac{(ax+b)^2}{2a}) - \frac{2n}{3} (ax+b)^2 {}_1F_1(-n+1; \frac{5}{2}, \frac{(ax+b)^2}{2a})} e^{-f^x} P(\tau) \, d\tau$
$-n e^{-x+f^x} P(\tau) \, d\tau$	$\frac{1}{x} e^{x-f^x} P(\tau) \, d\tau$	$-\frac{1}{nx} e^x \frac{{}_1F_1(-n; 1; x)}{{}_1F_1(-n+1; 2; x)} e^{-f^x} P(\tau) \, d\tau$
$-nbx^{c-1} e^{-bx+f^x} P(\tau) \, d\tau$	$\frac{1}{x^c} e^{bx-f^x} P(\tau) \, d\tau$	$\frac{c}{nx^c} e^{bx} \frac{{}_1F_1(-n; c; bx)}{{}_1F_1(-n+1; c+1; bx)} e^{-f^x} P(\tau) \, d\tau$
$nbx^{c-1} (x-1)^{(b+1-c-n)} e^{f^x} P(\tau) \, d\tau$	$x^{-c} (x-1)^{(c+n-b-1)} e^{-f^x} P(\tau) \, d\tau$	$\frac{c}{bn} x^{-c} (x-1)^{(c+n-b-1)} \frac{{}_1F_1(-n, b; c; x)}{{}_1F_1(-n+1, b+1; c+1; x)} e^{-f^x} P(\tau) \, d\tau$
$-n^2 x^{c-1} (x-1)^{(1-c-2n)} e^{f^x} P(\tau) \, d\tau$	$x^{-c} (x-1)^{(c+2n-1)} e^{-f^x} P(\tau) \, d\tau$	$-\frac{c}{n^2} x^{-c} (x-1)^{(c+2n-1)} \frac{{}_1F_1(-n, -n; c; x)}{{}_1F_1(-n+1, -n+1; c+1; x)} e^{-f^x} P(\tau) \, d\tau$
$n(n+1) e^{f^x} P(\tau) \, d\tau$	$\frac{1}{x^2-1} e^{-f^x} P(\tau) \, d\tau$	$\frac{2}{(x^2-1)} \frac{{}_2F_1(-n, n+1; 1; \frac{1-x}{2})}{{}_2F_1(-n+1, n+2; 2; \frac{1-x}{2})} e^{-f^x} P(\tau) \, d\tau$
$\frac{n(n+a+b+1)}{(x-1)^{-a}(x+1)^{-b}} e^{f^x} P(\tau) \, d\tau$	$\frac{e^{-f^x} P(\tau) \, d\tau}{(x-1)^{a+1}(x+1)^{b+1}}$	$\frac{(x-1)^{-a-1}}{(x+1)^{b+1}} \frac{{}_2F_1(-n, n+a+b+1; a+1; \frac{1-x}{2})}{{}_2F_1(-n+1, n+a+b+2; a+2; \frac{1-x}{2})} e^{-f^x} P(\tau) \, d\tau$
$\frac{n^2}{\sqrt{x^2-1}} e^{f^x} P(\tau) \, d\tau$	$\frac{e^{-f^x} P(\tau) \, d\tau}{\sqrt{x^2-1}}$	$\frac{e^{-f^x} P(\tau) \, d\tau}{n^2 \sqrt{x^2-1}} \frac{{}_2F_1(-n, n; \frac{1}{2}; \frac{1-x}{2})}{{}_2F_1(-n+1, n+1; \frac{3}{2}; \frac{1-x}{2})}$
$n(n+2) \sqrt{x^2-1} e^{f^x} P(\tau) \, d\tau$	$\frac{e^{-f^x} P(\tau) \, d\tau}{(x^2-1)^{3/2}}$	$\frac{3e^{-f^x} P(\tau) \, d\tau}{n(n+2)(x^2-1)^{3/2}} \frac{{}_2F_1(-n, n+2; \frac{3}{2}; \frac{1-x}{2})}{{}_2F_1(-n+1, n+3; \frac{5}{2}; \frac{1-x}{2})}$
$n(n+2k)(x^2-1)^{k-1/2} e^{f^x} P(\tau) \, d\tau$	$\frac{e^{-f^x} P(\tau) \, d\tau}{(x^2-1)^{1/2+k}}$	$\frac{(1+2k) e^{-f^x} P(\tau) \, d\tau}{n(n+2k)(x^2-1)^{k+1/2}} \frac{{}_2F_1(-n, n+2k; k+\frac{1}{2}; \frac{1-x}{2})}{{}_2F_1(-n+1, n+2k+1; k+\frac{3}{2}; \frac{1-x}{2})}$
$n(n+2k+1)(x^2-1)^k e^{f^x} P(\tau) \, d\tau$	$\frac{e^{-f^x} P(\tau) \, d\tau}{(x^2-1)^{1+k}}$	$\frac{2(1+k) e^{-f^x} P(\tau) \, d\tau}{n(n+2k+1)(x^2-1)^{k+1}} \frac{{}_2F_1(-n, n+2k+1; k+1; \frac{1-x}{2})}{{}_2F_1(-n+1, n+2k+2; k+2; \frac{1-x}{2})}$
$n(n+1) e^{-\frac{2}{x}+f^x} P(\tau) \, d\tau$	$\frac{1}{x^2} e^{\frac{2}{x}-f^x} P(\tau) \, d\tau$	$\frac{2e^{\frac{2}{x}-f^x} P(\tau) \, d\tau}{n(n+1)x^2} \frac{{}_2F_0(-n, n+1; -; -\frac{x}{2})}{{}_2F_0(-n+1, n+2; -; -\frac{x}{2})}$
$\frac{n(n+a-1)}{x^2-a} e^{-\frac{b}{x}+f^x} P(\tau) \, d\tau$	$\frac{1}{x^a} e^{\frac{b}{x}-f^x} P(\tau) \, d\tau$	$\frac{b e^{\frac{b}{x}-f^x} P(\tau) \, d\tau}{n(n+a-1)x^a} \frac{{}_2F_0(-n, n+a-1; -; -\frac{x}{b})}{{}_2F_0(-n+1, n+a; -; -\frac{x}{b})}$

Table 6. Closed-form solutions for the Riccati differential equation $y' + P(x)y + Q(x)y^2 = R(x)$ by theorem 5. Here $R(x)$ is an arbitrary differentiable function.

Riccati equation	Solution $y_n, n = 0, 1, 2, \dots$
$y' + \left(\frac{R'(x)}{R(x)} - 2x\right)y + \left(-\frac{4n}{R(x)} + \left(\frac{R'(x)}{R^2(x)} - \frac{2x}{R(x)}\right)'\right)y^2 = R(x)$	$\frac{R(x)}{-4nx \frac{{}_1F_1(-n+1; \frac{5}{2}; x^2)}{{}_1F_1(-n; \frac{1}{2}; x^2)} + \frac{R'(x)}{R(x)} - 2x}$
$y' + \left(\frac{R'(x)}{R(x)} - 2x\right)y + \left(-\frac{2(2n+1)}{R(x)} + \left(\frac{R'(x)}{R^2(x)} - \frac{2x}{R(x)}\right)'\right)y^2 = R(x)$	$-\frac{1}{x} + \frac{4nx}{3} \frac{{}_1F_1(-n+1; \frac{5}{2}; x^2)}{{}_1F_1(-n; \frac{3}{2}; x^2)} + \frac{R'(x)}{R(x)} - 2x$
$y' + \left(\frac{R'(x)}{R(x)} - ax - b\right)y + \left(-\frac{2na}{R(x)} + \left(\frac{R'(x)}{R^2(x)} - \frac{(ax+b)}{R(x)}\right)'\right)y^2 = R(x)$	$\frac{R(x)}{-2n(ax+b) \frac{{}_1F_1(-n+1; \frac{3}{2}; \frac{(ax+b)^2}{2a})}{{}_1F_1(-n; \frac{1}{2}; \frac{(ax+b)^2}{2a})} + \frac{R'(x)}{R(x)} - ax - b}$
$y' + \left(\frac{R'(x)}{R(x)} - ax - b\right)y + \left(-\frac{(2n+1)a}{R(x)} + \left(\frac{R'(x)}{R^2(x)} - \frac{(ax+b)}{R(x)}\right)'\right)y^2 = R(x)$	$-\frac{a}{ax+b} + \frac{2n}{3} \frac{(ax+b) \frac{{}_1F_1(-n+1; \frac{5}{2}; \frac{(ax+b)^2}{2a})}{{}_1F_1(-n; \frac{3}{2}; \frac{(ax+b)^2}{2a})} + \frac{R'(x)}{R(x)} - ax - b}$
$y' + \left(\frac{R'(x)}{R(x)} - b + \frac{c}{x}\right)y + \left(-\frac{nb}{xR(x)} + \left(\frac{R'(x)}{R^2(x)} - \frac{(b-\frac{c}{x})}{R(x)}\right)'\right)y^2 = R(x)$	$-\frac{nb}{c} \frac{{}_1F_1(-n+1; c+bx)}{{}_1F_1(-n; c; bx)} + \frac{R'(x)}{R(x)} - b + \frac{c}{x}$
$y' + \left(\frac{R'(x)}{R(x)} - \frac{(-2n+1)x-c}{x(1-x)}\right)y + \left(\frac{n^2}{x(1-x)R(x)} + \left(\frac{R'(x)}{R^2(x)} - \frac{(-2n+1)x-c}{x(1-x)R(x)}\right)'\right)y^2 = R(x)$	$\frac{R(x)}{\frac{n^2}{c^2} \frac{{}_2F_1(-n+1, -n+1; c+1, x)}{{}_2F_1(-n, -n; c; x)} + \frac{R'(x)}{R(x)} - \frac{(-2n+1)x-c}{x(1-x)}}$
$y' + \left(\frac{R'(x)}{R(x)} - \frac{(-n+b+1)x-c}{x(1-x)}\right)y + \left(\frac{-nb}{x(1-x)R(x)} + \left(\frac{R'(x)}{R^2(x)} - \frac{(-n+b+1)x-c}{x(1-x)R(x)}\right)'\right)y^2 = R(x)$	$\frac{R(x)}{\frac{-nb}{c^2} \frac{{}_2F_1(-n+1, b+1; c+1, x)}{{}_2F_1(-n, -n; c; x)} + \frac{R'(x)}{R(x)} - \frac{(-n+b+1)x-c}{x(1-x)}}$
$y' + \left(\frac{R'(x)}{R(x)} + \frac{(ax+b)}{x^2}\right)y + \left(\frac{n(n+a-1)}{x^2R(x)} + \left(\frac{R'(x)}{R^2(x)} + \frac{(ax+b)}{x^2R(x)}\right)'\right)y^2 = R(x)$	$\frac{R(x)}{\frac{n(n+a-1)}{b} \frac{{}_2F_0(-n+1, n+a; -; -\frac{x}{b})}{{}_2F_0(-n, n+a-1; -; -\frac{x}{b})} + \frac{R'(x)}{R(x)} + \frac{(ax+b)}{x^2}}$
$y' + \left(\frac{R'(x)}{R(x)} + \frac{2(k+1)x}{(1-x^2)}\right)y + \left(-\frac{n(n+2k+1)}{(1-x^2)R(x)} + \left(\frac{R'(x)}{R^2(x)} - \frac{2(k+1)x}{(1-x^2)R(x)}\right)'\right)y^2 = R(x)$	$\frac{R(x)}{\frac{n(n+2k+1)}{2(k+1)} \frac{{}_2F_1(-n+1, n+2k+2; k+2; \frac{1-x}{2})}{{}_2F_1(-n, n+2k+1; k+1; \frac{1-x}{2})} + \frac{R'(x)}{R(x)} - \frac{2(k+1)x}{(1-x^2)}}$
$y' + \left(\frac{R'(x)}{R(x)} + \frac{(2k+1)x}{(1-x^2)}\right)y + \left(-\frac{n(n+2k)}{(1-x^2)R(x)} + \left(\frac{R'(x)}{R^2(x)} - \frac{(2k+1)x}{(1-x^2)R(x)}\right)'\right)y^2 = R(x)$	$\frac{R(x)}{\frac{n(n+2k)}{(2k+1)} \frac{{}_2F_1(-n+1, n+2k+1; k+\frac{3}{2}; \frac{1-x}{2})}{{}_2F_1(-n, n+2k; k+\frac{1}{2}; \frac{1-x}{2})} + \frac{R'(x)}{R(x)} - \frac{(2k+1)x}{(1-x^2)}}$

An immediate result implied by this theorem is the following: for arbitrary $P(x)$, the Riccati equation

$$y' + P(x)y + e^{\int^x (\lambda_0(\tau) - P(\tau)) d\tau} y^2 = s_0(x) e^{-\int^x (\lambda_0(\tau) - P(\tau)) d\tau} - (P(x) e^{-\int^x (\lambda_0(\tau) - P(\tau)) d\tau})' \quad (33)$$

has the particular solution

$$y(x) = \left[-\frac{s_{n-1}(x)}{\lambda_{n-1}(x)} \right] e^{-\int^x (\lambda_0(\tau) - P(\tau)) d\tau} \quad (34)$$

4. Conclusion

It is well known that a Riccati equation can be transformed to a second-order linear differential equation by means of a suitable transformation. Using this fact along with a criterion, recently introduced, which guarantees the existence of polynomial solutions to second-order linear differential equations, we are able to derive analytic closed-form solutions for different classes of Riccati equation. By using the methods developed in this paper, the tables of solutions we present can easily be extended. For any given pair of differentiable functions, λ_0 and s_0 , satisfying the termination condition (2) along with (3), a corresponding class of exactly solvable Riccati equation can be generated.

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